4. Find all two-variable polynomials $p(x,y)$ with real coefficients such that $p(x+y,x-y)=2p(x,y)$ for all real numbers x and y.

Solution by Arkady Alt, San Jose, CA, USA.

Note that $p(2x,2y)=p((x+y)+(x-y),(x+y)-(x-y))=2p(x+y,x-y)=4p(x,y)$. Excluding the trivial case $p(x,y)\equiv 0$ we assume further that $p(x,y)\neq 0$.

Since $p(0,0)=p(2\cdot 0,2\cdot 0)=4p(0,0)$ then $p(0,0)=0$ and $p(x,y)$ is not constant, moreover $p(x,0)$ and $p(0,y)$ are not constants. Note that any such two-variable polynomial $p(x,y)$ can be represented in the form $p(x,y)=A(x)+B(y)+xyC(x,y)$, where $\deg A(x)=n>0, \deg B(y)=m>0,$ more precisely $A(x)=p(x,0)=a_nx^n+a_{n-1}x^{n-2}+\ldots+a_1x,\ B(y)=p(0,y)=b_mx^m+b_{m-1}x^{m-1}+\ldots+b_1x$, where $a_n\neq 0, b_m\neq 0.$

Since $p(2x,2y)=4p(x,y)$ then in particular for $y=0$ and any real x we have $p(2x,0)=4p(x,0)$ if and only if $2^na_n=4a_n, 2^{n-1}a_{n-1}=4a_{n-1},\ldots,2a_1=4a_1$ if and only if $n=2, k_1=0,$ Similarly we obtain $m=2, b_1=0.$

Thus, $p(x,y)=ax^2+xyC(x,y)+by^2.$ Since $p(x,x)=x^2(a+b+C(x,x))$ and $p(2x,2x)=4p(x,x)$ then for $x\neq 0$ we have $4x^2(a+b+C(2x,2x))=4x^2(a+b+C(x,x))$ if and only if $C(2x,2x)=C(x,x)$ if and only if $C(x,x)$ is constant.

Indeed, since $C(x,x)=c+c_1x^2+\cdots+c_kx^{2k}$ then $C(2x,2x)=C(x,x)$ if and only if $c_i=2^{2i}c_i, i=1,2,\ldots,k$ if and only if $c_i=0$, $i=1,2,\ldots,k.$ So, $p(x,y)=ax^2+cxy+by^2$ and since $p(x+y,x-y)=2p(x,y)$ if and only if $a(x+y)^2+c(x^2-y^2)+b(x-y)^2=2ax^2+2cxy+2by^2$ if and only if $(b+c-a)x^2+(a-b-c)y^2+2(a-c-b)xy=0$ for any x,y then $c=a-b.$

Therefore, $p(x,y)=ax^2+(a-b)xy+by^2$ and all such two-variable polynomials $p(x,y)$ of the second degree satisfy $p(x+y,x-y)=2p(x,y).$

Indeed, $p(x+y,x-y)=ax^2+2axy+ay^2+bx^2-2bxy+by^2+(b-a)(x^2-y^2)=2(ax^2+(a-b)xy+by^2)=2p(x,y).$

5. Let ω_1 and ω_2 be two circles such that the centre of ω_1 is located on ω_2. If the circles intersect at M and N, AB is an arbitrary diameter of ω_1, and A_1 and B_1 are the second intersections of AM and BN with the circle ω_2 (respectively), prove that A_1B_1 is equal to the radius of ω_1.

Solution by Michel Bataille, Rouen, France.

The following lemma will be used twice: Let C and C' be two circles intersecting at U,V. If points P,Q on C and P',Q' on C' are such that P,U,P' and Q,V,Q' are collinear, then PQ and $P'Q'$ are parallel.